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ON MINIMAX POSITION CONTROL*

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The problem of controlling a differential system with indeterminate noise is examined by the game-theoretic approach proposed and developed in /1—6/. The main result is the construction of the saddle point of the differential game being examined in the form of optimal mixed position strategies for a specific class of functionals designated as position functionals. It is established that the optimal strategies can be specified by functions depending solely on the current position and on a certain parameter the introduction of which is an essential element in the scheme being proposed. A stable approximation control scheme is constructed, guaranteeing the players results arbitrarily close to the game's value with a probability arbitrarily close to one if only the time step is sufficiently small.

1. We examine an object described by the differential equation

$$y^{\cdot} = f(t, y, u, v), \quad t_0 \leqslant t \leqslant \vartheta, \quad u \in P, \quad v \in Q, \quad |f(t, y, u, v)| \leqslant \varkappa\cdot(1 + |y|), \quad \varkappa = \text{const} \tag{1.1}$$

where y is an n-dimensional vector, t is time, u and v are vector-valued controls, P and Q are compacta. Function f is assumed to be continuous and to satisfy, in each bounded domain G, a Lipschitz condition in y with the constant L_G. We consider the motions starting in a prescribed bounded domain G_0. Then any motions on the interval $[t_0, \vartheta]$, encountered subsequently, do not leave some bounded domain G. All the continuous motions to be examined below satisfy a Lipschitz condition in t. Let us consider the problem of controls u and v that, respectively, minimize and maximize a functional γ prescribed on the motion $y_{t_0}[\cdot]_\vartheta = \{y[t], t_0 \leqslant t \leqslant \vartheta\}$. The result consists in the construction of optimal mixed strategies within the framework of a position differential game. In the scheme to be proposed a fundamental role is played by certain models.

2. The state of an x-model at instant t is characterized by an n-dimensional vector $x[t]$. A set of Borel-measurable functions $\{u_j[t] \in P,\ p_j[t] \geqslant 0,\ j = 1, 2, \ldots, N^{(1)};\ p_1[t] + \ldots + p_{N(1)}[t] = 1\}$ is called an action $F^{(1)}(t_*, t^*)$ of the first player on the interval $[t_*, t^*)$, $t_0 \leqslant t_* < t^* \leqslant \vartheta$. A set of measurable functions $\{v_k[t] \in Q,\ q_k[t] \geqslant 0,\ k = 1, 2, \ldots, N^{(2)};\ q_1[t] + \ldots + q_{N(2)}[t] = 1\}$ is called an action $F^{(2)}(t_*, t^*)$ of the second player. For a given initial position $\{t_*, x_*\}$ the actions $F^{(1)}(t_*, t^*)$ and $F^{(2)}(t_*, t^*)$ generate a motion $x[t]$, $t_* \leqslant t \leqslant t^*$, being an absolutely continuous solution of the equation

$$x^{\cdot}[t] = \sum_{j,k=1}^{N(1),\,N(2)} f(t, x[t], u_j[t], v_k[t])\cdot p_j[t]\cdot q_k[t] \tag{2.1}$$

An action $F^{(1)}(t_*, t^*)(F^{(2)}(t_*, t^*))$ is said to be elementary if $u_j[t] = u_j = \text{const}$, $p_j[t] = p_j = \text{const}$ $(v_k[t] = v_k = \text{const},\ q_k[t] = q_k = \text{const})$.

A rule that from any possible values $\{t, x, \varepsilon > 0\}$ fixes the constant vectors $\{p_1, \ldots, p_{N(1)}\}$, $\{u_1, \ldots, u_{N(1)}\}$, $u_j \in P$ $(\{q_1, \ldots, q_{N(2)}\}, \{v_1, \ldots, v_{N(2)}\}, v_k \in Q)$ is called a strategy U_x on the first player (V_x of the second player). We construct the $\{\varepsilon, \Delta\}$-motion of the x-model, generated by strategy U_x from position $\{t_*, x[t_*]\}$ in steps $[\tau_i, \tau_{i+1}]$. Suppose that a partitioning Δ of interval $[t_0, \vartheta]$ by points τ_i has been chosen such that $t_0 = \tau_0 < \tau_1 < \ldots < \tau_m = \vartheta$. We fix $\varepsilon > 0$. If a position $\{\tau_i, x[\tau_i]\}$ is realized, then from $\{\tau_i, x[\tau_i], \varepsilon\}$ the strategy U_x fixes the vectors $\{p_j^{[i]}\}$ and $\{u_j^{[i]}\}$ and the corresponding elementary action $F^{(1)}(\tau_i, \tau_{i+1})$ operates on the interval (τ_i, τ_{i+1}). The second player can choose any action $F^{(2)}(\tau_i, \tau_{i+1})$ in (2.1). This pair of actions realizes the motion $x_\Delta^\varepsilon[t]$, $\tau_i \leqslant t \leqslant \tau_{i+1}$, i.e., a solution of Eq. (2.1) for $t_* = \tau_i$, $t^* = \tau_{i+1}$, $u_j[t] = u_j^{[i]}$, $p_j[t] = p_j^{[i]}$. The motion of the x-model, generated by strategy V_x is defined analogously.

3. Let us examine the functionals $\gamma(x_{t_*}[\cdot]_\vartheta)$, $t_0 \leqslant t_* \leqslant \vartheta$, $x_{t_*}[\cdot]_\vartheta = \{x[t], t_* \leqslant t \leqslant \vartheta\}$, defined on piecewise-continuous curves $x_{t_*}[\cdot]_\vartheta$ having only a finite number of points of discontinuity of the first kind and being right-continuous. The functionals are continuous in the following sense: for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that $|\gamma(x_{t_*}^{(1)}[\cdot]_\vartheta) - \gamma(x_{t_*}^{(2)}[\cdot]_\vartheta)| < \varepsilon$ as soon as

$$\sup_{t_* \leqslant t \leqslant \vartheta} |x^{(1)}[t] - x^{(2)}[t]| < \delta, \quad x^{(i)}[t] \in G$$

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Functionals representable in the form

$$\gamma(x_{t_*}[\cdot]_\vartheta) = \varphi(x_{t_*}[\cdot]_{t^*}, \alpha), \quad \alpha = \gamma(x_{t^*}[\cdot]_\vartheta) \tag{3.1}$$

where the function $\varphi(x_{t_*}[\cdot]_{t^*}, \alpha)$, is continuous and does not decrease with respect to α when curve $x_{t_*}[\cdot]_{t^*}$ is fixed, are called position functionals. In particular,

$$\gamma = \int_{t_0}^{\vartheta} w(t, x[t])\, dt + \sigma(x[\vartheta]), \quad \gamma = \inf_{t_0 \leqslant \tau \leqslant \vartheta} \max(\sup_{t_0 \leqslant t \leqslant \tau} \omega(t, x, [t]), \sigma(\tau, x[\tau]))$$

are such functionals. If function $x_{t_*}[\cdot]_\vartheta$ is discontinuous at $t = t^*$, then in the notation $\varphi(x_{t_*}[\cdot]_{t^*}, \alpha)$ the symbol $x_{t_*}[\cdot]_{t^*}$ denotes the function $x[t]$ for $t_* \leqslant t < t^*$.

4. Let $X(U, \varepsilon, \delta, t_*, x_*) = \{x_{t_*}[\cdot]_\vartheta\}$ $(X(V, \varepsilon, \delta, t_*, x_*) = \{x_{t_*}[\cdot]_\vartheta\})$ be a bundle of $\{\varepsilon, \Delta\}$-motions generated by strategy U_x (V_x) from position $\{t_*, x_*\}$, $t_* \in [t_0, \vartheta]$ when $(\tau_{i+1} - \tau_i) \leqslant \delta$. Let

$$\inf_U \inf_\varepsilon \inf_\delta \sup[\gamma(x_{t_*}[\cdot]_\vartheta),\ x_{t_*}[\cdot]_\vartheta \in X(U, \varepsilon, \delta, t_*, x_*)] = \gamma^{(1)}(t_*, x[t_*])$$

$$\sup_V \sup_\varepsilon \sup_\delta \inf[\gamma(x_{t_*}[\cdot]_\vartheta),\ x_{t_*}[\cdot]_\vartheta \in X(V, \varepsilon, \delta, t_*, x_*)] = \gamma^{(2)}(t_*, x[t_*])$$

Strategy U_x° is said to be optimal if for any $\zeta > 0$

$$\gamma(x_{t_*}[\cdot]_\vartheta) \leqslant \gamma^{(1)}(t_*, x[t_*]) + \zeta, \quad x_{t_*}[\cdot]_\vartheta \in X(U_x^\circ, \varepsilon, \delta, t_*, x_*), \ \varepsilon \leqslant \varepsilon(\zeta), \ \delta \leqslant \delta(\varepsilon)$$

We note that for any preselected $\zeta > 0$ we have

$$\gamma(x_{t_0}[\cdot]_\vartheta) \leqslant \varphi(x_{t_0}[\cdot]_{t_*}, \gamma^{(1)}(t_*, x[t_*])) + \zeta$$

where $x_{t_0}[\cdot]_{t_*}$ is the motion realized by the instant t_* and $x_{t_0}[\cdot]_\vartheta$ is the motion composed of $x_{t_0}[\cdot]_{t_*}$ and any $\{\varepsilon, \Delta\}$-motion $x_{t_*}[\cdot]_\vartheta \in X(U_x^\circ, \varepsilon, \delta, t_*, x[t_*])$ with $\varepsilon \leqslant \varepsilon(\zeta)$, $\delta \leqslant \delta(\varepsilon, \zeta)$. Strategy V_x° is said to be position-optimal if for any $\zeta > 0$

$$\gamma(x_{t_*}[\cdot]_\vartheta) \geqslant \gamma^{(2)}(t_*, x[t_*]) - \zeta, \quad x_{t_*}[\cdot]_\vartheta \in X(V_x^\circ, \varepsilon, \delta, t_*, x[t_*]), \quad \varepsilon \leqslant \varepsilon(\zeta), \quad \delta \leqslant \delta(\varepsilon, \zeta).$$

We have

$$\gamma(x_{t_0}[\cdot]_\vartheta) \geqslant \varphi(x_{t_0}[\cdot]_{t_*}, \gamma^{(2)}(t_*, x[t_*])) - \zeta$$

where $x_{t_0}[\cdot]_{t_*}$ is the motion realized by the instant t_* and $x_{t_0}[\cdot]_\vartheta$ is the motion composed of $x_{t_0}[\cdot]_{t_*}$ and any $\{\varepsilon, \Delta\}$-motion $x_{t_*}[\cdot]_\vartheta \in X(V_x^\circ, \varepsilon, \delta, t_*, x[t_*])$, $\varepsilon \leqslant \varepsilon(\zeta)$, $\delta \leqslant \delta(\varepsilon, \zeta)$.

We say that the pair $\{U_x^\circ, V_x^\circ\}$ constitutes a position saddle point and yields the game's value $\gamma^\circ(t, x)$ if

$$\gamma^{(1)}(t, x) = \gamma^{(2)}(t, x) = \gamma^\circ(t, x), \quad \gamma^\circ(t_*, x[t_*]) \geqslant \varphi(x_{t_*}[\cdot]_{t^*}, \gamma^\circ(t^*, x[t^*])) - \zeta$$

$$x_{t_*}[\cdot]_\vartheta \in X(U_x^\circ, \varepsilon, \delta, t_*, x[t_*]), \quad \gamma^\circ(t_*, x[t_*]) \leqslant \varphi(x_{t_*}[\cdot]_{t^*}, \gamma^\circ(t^*, x[t^*])) + \zeta$$

$$x_{t_*}[\cdot]_\vartheta \in X(V_*^\circ, \varepsilon, \delta, t_*, x[t_*])$$

for any $\zeta > 0$ when $\varepsilon \leqslant \varepsilon(\zeta)$ and $\delta \leqslant \delta(\varepsilon, \zeta)$. The problem is to construct the strategies $\{U_x^\circ, V_x^\circ\}$ constituting the saddle point.

5. The state of the w-model is characterized by the n-dimensional vector $w[t]$. The motions of the w-model are generated by actions defined in the same way as for the x-model and which are marked by asterisks, $F_*^{(1)}(t_*, t^*)$ and $F_*^{(2)}(t_*, t^*)$, to distinguish them from the actions of the x-model. A motion of the w-model on interval $[t_*, t^*]$ is a solution of

$$w^{\cdot}[t] = \sum_{j, k=1}^{N^{(1)}, N^{(2)}} f(t, w[t], u_j[t], v_k[t]) \cdot p_{*j}[t] \cdot q_{*k}[t], \quad w[t_*] = w_* \tag{5.1}$$

A rule which at an instant $\tau_i \geqslant t_*$ fixes $\tau_{i+1} > \tau_i$ and $F_{*i}^{(2)} = F_*^{(2)}(\tau_i, \tau_{i+1})$ on the basis of given $\{\tau_i, F_{*s}^{(1)}, F_{*s}^{(2)}, s = 0, 1, \ldots, i-1\}$, $F_{*s}^{(k)} = F_*^{(k)}(\tau_s, \tau_{s+1})$ is called a $Q_{\{t_*, w_*\}}$-procedure. The first player fixes $F_*^{(1)} = F_*^{(1)}(\tau_i, \tau_{i+1})$ from the known τ_{i+1} and $F_{*i}^{(2)}$. In the notation $Q_{\{t_*, w_*\}}$ the symbol $\{t_*, w_*\}$ indicates that the motion is formed from the position $\{t_*, w_*\}$. Procedures in which the number of instants τ_i from $\tau_0 = t_*$ to ϑ is finite for each possible realization of $w_{t_*}[\cdot]_\vartheta$ are admissible. This number can be different for different realizations. Realizations with any number of instants τ_i are possible.

Let β be some number. A procedure $Q_{\{t_*, w_*\}}$ is called a $(\beta - Q_{\{t_*, w_*\}})$-procedure if for any motion $w_{t_*}[\cdot]_\vartheta$ generated by it the condition $\gamma(w_{t_*}[\cdot]_\vartheta) > \beta$ is fulfilled. By $\rho(t, w)$ we denote the least upper bound of the β for which a $(\beta - Q_{\{t_*, w_*\}})$-procedure exists. It can be shown that function $\rho(t_*, w_*)$ is continuous in w. The following statements are valid.

Lemma 5.1. Let a position $\{t_*, w_*\}$ be given and let $t^* > t_*$, $\varepsilon_* > 0$ and $F_*^{(2)}(t_*, t^*)$ be specified. Then we can find $\varepsilon^* > 0$ and an action $F_*^{(1)}(t_*, t^*)$ which in pair with $F_*^{(2)}(t_*, t^*)$ generates a motion $w_{t_*}[\cdot]_{t^*}$ such that

$$\varphi\,(w_{t_*}[\cdot]_{t^*},\ \rho\,(t^*, w^*) + 2\varepsilon^*) \geqslant \rho\,(t_*, w_*) + 2\varepsilon_* \tag{5.2}$$

Lemma 5.2. Let a position $\{t_*, w_*\}$ be given and let $t^* > t_*$, $\varepsilon_* > 0$ and a motion $w_{t_*}[\cdot]_{t^*}$ generated by a $(\beta - Q_{\{t_*, w_*\}})$-procedure with $\beta = \rho\,(t_*, w_*) - \varepsilon_*$ be specified. Then we can find $\varepsilon^* > 0$ such that

$$\varphi\,(w_{t_*}[\cdot]_{t^*},\ \rho\,(t^*, w^*) - 2\varepsilon^*) \geqslant \rho\,(t_*, w_*) - 2\varepsilon_*, \quad w^* = w_{t_*}[t^*]_{t^*} \tag{5.3}$$

Consider the function

$$\lambda\,(t, x, w) = |\,x - w|^2 \exp\{-3L_G \cdot (t - t_0)\} \tag{5.4}$$

Lemma 5.3. Let a bounded domain G^* be specified in space $\{x\}$ and let $\varepsilon > 0$. Then we can find vectors $\{u_1, \ldots, u_N\}$, $\{v_1, \ldots, v_N\}$, $u_j \in P$, $v_k \in Q$, $N = N\,(\varepsilon, G^*)$, and a number $\delta\,(\varepsilon, G^*) > 0$ such that if the vectors $\{p_1^\circ, \ldots, p_N^\circ\}$ and $\{q_{*1}^\circ, \ldots, q_{*N}^\circ\}$ have been defined from the conditions

$$\max_q \sum_{j,k=1}^{N,N} \langle s_* \cdot f\,(t_*, x_*, u_j, v_k) \cdot p_j^\circ \cdot q_k \rangle = \min_p \operatorname{Idem}(p_j^\circ \to p_j) \tag{5.5}$$

$$\min_{p_*} \sum_{j,k=1}^{N,N} \langle s_* \cdot f\,(t_*, w_*, u_j, v_k) \cdot p_{*j} \cdot \mathring{q}_{*k} \rangle = \max_{q_*} \operatorname{Idem}(\mathring{q}_{*k} \to q_{*k}) \tag{5.6}$$

where $s_* = x_* - w_*$, for $t_* \in [t_0, \vartheta]$, $x_* \in G^*$ and $w_* \in G^*$, then for $t_* \leqslant t \leqslant t_* + \delta\,(\varepsilon, G^*)$ the elementary action $F^{(1)}(t_*, t)$, corresponding to $\{p_j^\circ\}$, in pair with any action $F^{(2)}(t_*, t)$, and the elementary action $F_*^{(2)}(t_*, t)$, corresponding to $\{q_{*k}^\circ\}$, in pair with any action $F_*^{(1)}(t_*, t)$, generate motions $x\,[t]$ and $w\,[t]$ for which

$$\lambda\,(t, x\,[t], w\,[t]) \leqslant \lambda\,(t_*, x_*, w_*) + \varepsilon \cdot (t - t_*) \tag{5.7}$$

Here and further the symbol Idem in the right hand side of an equality denotes an expression coinciding with the left hand side of this equality with the change of symbols indicated within the parentheses.

Lemma 5.4. This lemma can be stated analogously to Lemma 5.3 by interchanging the symbols p and q, as well as the actions $F^{(1)}(t_*, t)$ and $F^{(2)}(t_*, t)$.

6. We construct strategies U_x^e and V_x^e which we call extremal. We select $\varepsilon > 0$ and $\{t_*, x_*\}$. By K_* we denote the set of points w satisfying the inequality

$$\lambda\,(t_*, x_*, w) \leqslant \varepsilon \cdot (t_* - t_0) \tag{6.1}$$

A point $w_* \in K_*$ is called an accompanying point if it satisfies the condition:

a) for constructing strategy U_x^e

$$\rho\,(t_*, w_*) = \min_{w \in K_*} \rho\,(t_*, w) \tag{6.2}$$

b) for constructing strategy V_x^e

$$\rho\,(t_*, w_*) = \max_{w \in K_*} \rho\,(t_*, w) \tag{6.3}$$

There can be several such points w_*. We choose one of them for each given $\{t_*, x_*, \varepsilon\}$. We fix $\varepsilon^* > 0$ and we select domain G^* such that $w_* \in G^*$ for all $\varepsilon < \varepsilon^*$ and $x_* \in G$. Later we select only $\varepsilon < \varepsilon^*$.

An extremal strategy $U_x^e\,(V_x^e)$ is a rule that associates with possible values of $\{t_*, x_*, \varepsilon\}$ the vectors $\{u_j\}$ and $\{p_j^\circ\}$ ($\{v_k\}$ and $\{q_k^\circ\}$) connected by condition (5.5) (corresponding to the condition from Lemma 5.4) and satisfying estimate (5.7) wherein w_* is the accompanying point. As a consequence of the measurable selector theorem /7/ the vectors $\{p_j^\circ\}$ and $\{q_k^\circ\}$ can be taken as functions Borel-measurable in x_*, since conditions (5.5), (5.7), the corresponding conditions from Lemma 5.4, and relations (6.1)—(6.3) define for the choice of $\{p_j^\circ\}$ and $\{q_k^\circ\}$ compact sets that are semicontinuous relative to x_*.

Suppose that the first player is guided by strategy U_x^e and has chosen $\varepsilon > 0$ and the partitioning $\Delta = \{\tau_i\}$. These data generate a certain motion $x^{\varepsilon}_{\Delta t_*}[\cdot]_\vartheta$. Let $w\,[\tau_i]$ be the accompanying point for $x\,[\tau_i]$. With motion $x_\Delta^\varepsilon[t]$ we associate an accompanying motion of the w-model. On the half-open interval $\tau_i \leqslant t < \tau_{i+1}$ this is a solution of Eq. (5.1) with boundary condition $\{\tau_i, w\,[\tau_i]\}$, generated by actions $F_*^{(2)}(\tau_i, \tau_{i+1})$ and $F_*^{(1)}(\tau_i, \tau_{i+1})$ from Lemma 5.3 with $x_* = x\,[\tau_i]$ and $w_* = w\,[\tau_i]$. Motion $w_{t_*}[\cdot]_\vartheta$ can undergo discontinuities at instants τ_i. We denote $w^{[i]} = \lim w\,[t]$, $t \to \tau_i - 0$.

Relying on Lemma 5.3 and on the properties of the extremal strategy we can show that for any $F^{(2)}(\tau_i, \tau_{i+1})$ and $F_*^{(1)}(\tau_i, \tau_{i+1})$ the estimate $\lambda\,(t, x_\Delta^\varepsilon[t], w\,[t]) \leqslant \varepsilon + \varepsilon \cdot (t - t_*)$, $t_* \leqslant t \leqslant \vartheta$, holds if $\max|\,\tau_{i+1} - \tau_i\,| < \delta\,(\varepsilon, G)$. Moreover, $\rho\,(\tau_{i+1}, w\,[\tau_{i+1}]) \leqslant \rho\,(\tau_{i+1}, w^{[i+1]})$. In the accompanying motion $w\,[t]$ let the action $F_*^{(1)}(\tau_i, \tau_{i+1})$ at each step be chosen in accord with Lemma 5.1. Then at each step we have

$$\varphi\,(w_{\tau_i}[\cdot]_{\tau_{i+1}}, \rho\,(\tau_{i+1}, w^{[i+1]}) + 2\varepsilon_{i+1}) \leqslant \rho\,(\tau_i, w\,[\tau_i]) + 2\varepsilon_i$$

whence we obtain $\gamma(w_{\tau_i}[\cdot]_\vartheta) \leqslant \rho(\tau_i, w[\tau_i]) + 2\varepsilon_i$ and, setting $\varepsilon_0 = \varepsilon$, by induction we then have

$$\gamma(w_{t_*}[\cdot]_\vartheta) \leqslant \rho(t_*, w_*) + 2\varepsilon \tag{6.4}$$

Using the proximity of motion $x^\varepsilon{}_{\Delta^{t_*}}[\cdot, U_x{}^e]_\vartheta$ to motion $w_{t_*}[\cdot]_\vartheta$, the continuity of functional γ and the continuity of $\rho(t, w)$ in w, we obtain the following result.

Theorem 6.1. Let position $\{t_*, x_*\}$ be realized. If, beginning with some instant t_*, the first player uses the extremal strategy $U_x{}^e$, then for any arbitrarily small ε and sufficiently small step $\delta(\varepsilon)$ we have

$$\gamma(x^\varepsilon_{\Delta t_*}[\cdot]_\vartheta) \leqslant \rho(t_*, x_*) + \xi(\varepsilon), \quad \lim_{\varepsilon\to 0} \xi(\varepsilon) = 0 \tag{6.5}$$

for any $\{\varepsilon, \Delta\}$-motion $x^\varepsilon_{\Delta t_*}[\cdot]_\vartheta$, $x^\varepsilon_{\Delta t_*}[t_*]_\vartheta = x_*$.

Now let the second player be guided by the extremal strategy V^e_x. Rely-on the properties of this strategy and on Lemmas 5.2 and 5.4, we can derive the relations $\lambda(t, x^\varepsilon_\Delta[t], w[t]) \leqslant \varepsilon + \varepsilon\cdot(t-\tau_i)$, $\tau_i \leqslant t \leqslant \tau_{i+1}$, $\rho(\tau_{i+1}, w[\tau_{i+1}]) \geqslant \rho(\tau_{i+1}, w^{[i+1]})$ and then

$$\gamma(w_{t_*}[\cdot]_\vartheta) \geqslant \rho(t_*, w_*) - 2\varepsilon \tag{6.6}$$

where $\{t_*, w_* = w[t_*]\}$ is the accompanying point for position $\{t_*, w_*\}$, $w_{t_*}[\cdot]_\vartheta$ is the motion of the w-model, which at each step is generated by the action $F^{(1)}_*(\tau_i, \tau_{i+1})$ chosen from the condition of Lemma 5.4, corresponding to condition (5.6), and by the action $F^{(2)}_*(\tau_i, \tau_{i+1})$ fixed by a $(\beta - Q_{(\tau_i, w[\tau_i])})$-procedure with $\beta = \rho(\tau_i, w[\tau_i]) - \varepsilon_i$. From this we obtain the following result.

Theorem 6.2. Let position $\{t_*, x_*\}$ be realized. If, beginning with some instant t_*, the second player uses the extremal strategy $V_x{}^e$, then for any arbitrarily small ε and sufficiently small step $\delta(\varepsilon) > 0$ we have the estimate

$$\gamma(x^\varepsilon_{\Delta t_*}[\cdot]_\vartheta) \geqslant \rho(t_*, x_*) - \eta(\varepsilon), \quad \lim_{\varepsilon\to 0} \eta(\varepsilon) = 0 \tag{6.7}$$

for any $\{\varepsilon, \Delta\}$-motion.

A comparison of Theorems 6.1 and 6.2 leads to the next statement.

Theorem 6.3. The extremal strategy U^e_x is the first player's optimal strategy. The extremal strategy V^e_x is the second player's optimal strategy. The strategy pair $\{U^e_x, V^e_x\}$ forms a position saddle point. The game's value is $\gamma^\circ(t, x) = \rho(t, x)$.

7. Let us now consider the original problem on the control of the given object (1.1). Using the control-with-guide method /1/, wherein the x-model is selected as the guide we define a certain united strategy U of the first player.

The motion of the object, generated from position $\{t_0, y_0 \in G_0\}$ by strategy U, is constructed simultaneously with some $\{\varepsilon, \Delta\}$-motion of the x-model, generated by some strategy U_x included in U. Suppose that U, $\varepsilon > 0$ and $\Delta = \{\tau_i\}$ have been chosen. From the data $\{y[\tau_i], x[\tau_i], \tau_i, \varepsilon\}$ some strategy U_y included in U fixes the elementary action $F^{(1)}_y(\tau_i, \tau_{i+1})$. A random test is carried out, corresponding to the probability distribution $\{p_1^{[i]}, \ldots, p_{N(1)}^{[i]}\}$ of the random variable $\{u_1, \ldots, u_{N(1)}\}$, defined by the action $F^{(1)}_y(\tau_i, \tau_{i+1})$. This test's result $u_{[i]}$ is the control for $\tau_i < t < \tau_{i+1}$. The second player, using some random mechanism of his own, develops a function $v_{[i]}[t]$, $\tau_i < t < \tau_{i+1}$. The object's motion on interval $[\tau_i, \tau_{i+1}]$ is a solution of the equation

$$\dot{y}[t] = f(t, y[t], u_{[i]}, v_{[i]}[t]) \tag{7.1}$$

The motion $x^\varepsilon_\Delta[t]$ is realized by the same partitioning $\Delta = \{\tau_i\}$ from some suitable position $\{t_0, x_0\}$. In the real control of the united system made up of the y-object and the x-model only the real first player controls the motion of the x-model. However, in connection with the material from Sects. 1–6 it is convenient to separate his action into the action of a fictitious first player who fixes $F^{(1)}(\tau_i, \tau_{i+1})$ in accordance with the selected strategy U_x and the action of a fictitious second player who fixes $F^{(2)}(\tau_i, \tau_{i+1})$ in accordance with some rule R_U. Thus, the real first player has at his disposal the collection $\{U_y, U_x, R_U, x_0\}$ which is called his united strategy U. The motion $\{y[t], x[t]\}$ generated by U is denoted $\{y_{t_0}[\cdot, U]_\vartheta, x_{t_0}[\cdot, U]_\vartheta\}$. It is obtained as a random motion since the choice of the forces is determined by random tests. The motions of the object and of the x-model, generated by strategies V_y and V_x and by some rule R_V, and the united strategy V of the second player are determined in similar fashion. The second player, of course, uses his own x-model.

Let us describe the construction of the extremal strategy U^e. We assume that the functions $u[\tau_i]$ and $v[t]$ are stochastically independent for $\tau_i \leqslant t \leqslant \tau_{i+1}$. The process $\{y[t], x[t]\}$ being analyzed here is formalized as a probabilistic process within the framework of the strict concepts of probability theory. This raises no principal difficulties since all the constructions carried out yield control functions measurable with respect to $\{y[\tau_i], x[\tau_i]\}$. We choose the initial condition x_0 from the condition

$$\lambda\,(t_0, x_0, y_0) \leqslant \varepsilon \tag{7.2}$$

The x-model's motion $x[t]$ is determined by the extremal strategy U_x^e. As the rule R_U^e we take one which fixes the actions $F^{(2)}(\tau_i, \tau_{i+1})$ defined by the condition

$$\min_p \sum_{j,k=1}^{N,N} \langle g^{[i]}\cdot f(\tau_i, x[\tau_i], u_j, v_k)\cdot p_j\cdot q_k^\circ\rangle = \max_q \operatorname{Idem}(q_k^\circ \to q_k), \quad g^{[i]} = y[\tau_i] - x[\tau_i] \tag{7.3}$$

The rule by which the elementary action $F_y^{(1)}(t_*, t) = \{u_j, p_j^\circ\}$ is fixed from the quantities $\{y_*, x_*, t_*, \varepsilon\}$, $y_* \in G^*$, $x_* \in G^*$, by the condition

$$\max_q \sum_{j,k=1}^{N,N} \langle g_*\cdot f(t_*, y_*, u_j, v_k)\cdot p_j^\circ\cdot q_k\rangle = \min_p \operatorname{Idem}(p_j^\circ \to p_j), \quad g_* = y_* - x_* \tag{7.4}$$

is called the extremal strategy U_y^e. The extremal strategy V_y^e is defined analogously. The strategy $U^e = \{U_y^e, U_x^e, R_U^e, x_0\}$ is called the united extremal strategy. The united extremal strategy V^e is constructed analogously. The following statement is valid.

Theorem 7.1. For any numbers $\zeta > 0$ and $0 \leqslant \chi < 1$ we can always select $\varepsilon > 0$ and $\delta(\varepsilon, G^*) > 0$ such that for all motions generated by $U^e = \{U_y^e, U_x^e, R_U^e, x_0\}$ the estimate

$$P(\gamma(y_{t_0}[\cdot, U^e]_\theta) \leqslant \rho(t_0, y_0) + \zeta) > \varkappa, \quad P(\gamma(y_{t_*}[\cdot, U^e]_\theta) \leqslant \rho(t_*, y[t_*]) + \zeta) > \varkappa \tag{7.5}$$

$$P(\varphi(y_{t_0}[\cdot, U^e]_{t_*}, \rho(t, y[t_*])) \leqslant \rho(t_0, y) + \zeta) > \varkappa$$

will hold for all motions generated by $V^e = \{V_y^e, V_x^e, R_V^e, x_0\}$ and estimates analogous to (7.5) will hold with U^e replaced by V^e and ζ by $-\zeta$, if only $\max_i |\tau_{i+1} - \tau_i| < \delta(\varepsilon, G^*)$. By P we have denoted the probability of the corresponding event. We observe that, by definition, from these conditions follow as well the fulfillment of conditions

$$P(\gamma(y_{t_0}[\cdot, U^e]_\theta) \leqslant \varphi(y_{t_0}[\cdot]_{t_*}, \rho(t_*, y[t_*])) + \zeta) > \varkappa, \quad P(\gamma(y_{t_0}[\cdot, V^e]_\theta) \geqslant \varphi(y_{t_0}[\cdot]_{t_*}, \rho(t_*, y[t_*])) - \zeta) > \varkappa \tag{7.6}$$

Thus the relations (7.5) and (7.6) obtained permit us to call the quantity $\rho(t, y)$ the value of the game and to call the united extremal strategies U^e and V^e described optimal strategies yielding the game's saddle point.

8. The fundamental Theorem 7.1 has been proved under the assumptions that functions $u[t]$ and $v[t]$ are stochastically independent for $\tau_i < t < \tau_{i+1}$. If the question is of control in a game with Nature, this independence condition can be adopted as a separate postulate. Without a logical contradiction this postulate imposes a constraint on the unknown mechanisms forming the noise $v[t]$. However, if we treat the process as a game between two real players each of whom can act on their own strategies U and V with their own partitionings $\Delta_U = \{\tau_i^U\}$ and $\Delta_V = \{\tau_i^V\}$, then one cannot adopt such an independence condition as a postulate. The connection between $u[t]$ and $v[t]$ depends upon the strategies U and V and the partitionings Δ_U and Δ_V chosen. However, this difficulty can be overcome in the following well-known manner. We assume that the forces $u[t] = u[\tau_i^U]$, $\tau_i^U < t < \tau_{i+1}^U$ and $v[t] = v[\tau_i^V]$, $\tau_i^V < t < \tau_{i+1}^V$, on object (1.1) are obtained as the results of random tests with probability distributions $\{p_j^\circ\}$ and $\{q_k^\circ\}$, which now correspond not to the values $y[\tau_i^U]$ and $y[\tau_i^V]$ but to the values $y[\tau_i^U - \tau^U]$ and $y[\tau_i^V - \tau^V]$, where $\tau^U > 0$ and $\tau^V > 0$ are constant information lags. Then, relying once again on the results in Sects.1— 7, we obtain the following statement.

Theorem 8.1. The united extremal strategies U^e and V^e, described in Sect.7 but developed on the basis of the lagging values $y[\tau_i^U - \tau^U]$ and $y[\tau_i^V - \tau^V]$, constitute the saddle point $\{U^\circ, V^\circ\}$ of the game being analyzed, with value $\gamma^\circ = \rho(t, y)$, i.e., thev ensure the fulfillment of conditions (7.2)— (7.5) if only the conditions

$$\tau_{i+1}^U - \tau_i^U < \delta^U < \tau^V, \quad \tau_{i+1}^V - \tau_i^V < \delta^V < \tau^U \tag{8.1}$$

are fulfilled, where $\delta^U, \tau^U, \delta^V, \tau^V$ are sufficiently small positive numbers.

It is important to note here that now both players simultaneously form one and the same motion $x[t]$, each using his own x-model and each forming this motion on the basis of his own united strategy U° or V°, and his own partitioning Δ_U or Δ_V.

9. The fundamental results given in Theorems 7.1 and 8.1 remain in force under conditions on function f somewhat more general than those in Sect.1. To be precise, we assume that this function is bounded in any bounded domain G, is continuous in y, u, v for fixed t, and is Borel-measurable in t for fixed y, u, v. We consider the equation

$$x[\tau] = x[t_*] + \int_{t_*}^{\tau}\int_P\int_Q f(t, x[t], u, v)\cdot\eta(dv, du \mid t)\,dt \tag{9.1}$$

where $\eta(dv, du \mid t)$ is a probability measure on $P \times Q$, weakly Borel-measurable in t. Let G

be a bounded domain. We assume that Eq. (9.1) has a unique solution for every $x[t_*] \in G$ and every measure η, which for all $t_0 \leqslant t \leqslant \vartheta$ is contained in some bounded domain G_G^*. All these conditions are obviously fulfilled under the conditions in Sect.1. However, under these more general conditions we can repeat all the lemmas and theorems from Sects.1—8, merely replacing in them the function λ of (5.4) by a certain function λ constructed in accordance with the ideas in /6/ and, when choosing the extremal vectors $\{p_j^\circ\}$ and $\{q_k^\circ\}$, replacing the function f by a suitable continuous function f_ε which approximates function f in a sufficiently large domain G^*, such that

$$\int_{t_0}^{t} \max_{y \in G, u, v} |f_\varepsilon(\tau, y, u, v) - f(\tau, y, u, v)| \, d\tau \leqslant \zeta[t], \quad \xi[t] < \varepsilon \cdot (\vartheta - t_0)$$

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REFERENCES

1. KRASOVSKII, N. N. and SUBBOTIN, A. I., Position Differential Games. Moscow, "Nauka", 1974.

2. KURZHANSKII, A. B., Control and Observation Under Indeterminacy Conditions. Moscow,"Nauka", 1977.

3. OSIPOV, Iu. S., Position control in parabolic systems. PMM Vol.41, No.2, 1977.

4. BATUHTIN, V. D., A programmed construction for the positional control. Lecture Notes in Computer Science, Vol.27. Berlin-Heidelberg-New York, Springer-Verlag, 1975.

5. CHENTSOV, A. G., On a guidance game problem. Dokl. Akad. Nauk SSSR, Vol.226, No.1, 1976.

6. KRIAZHIMSKII, A. V., On the theory of position differential games of approach-evasion. Dokl. Akad. Nauk SSSR, Vol.239, No.4, 1978.

7. IOFFE, A. D. and TIKHOMIROV, V. M., Theory of Extremal Problems. Moscow. "Nauka", 1974.

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